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Inter-World Transitions

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Inter-World Transitions

Droste Effects, Möbius Transformations, and Non-Euclidean Topologies

1. Abstract

This report presents the initial formulation of an exploratory research project investigating the use of conformal mappings (particularly Möbius transformations on the Riemann sphere) as mechanisms for creating visual and topological transitions between virtual worlds represented as omnidirectional 360° imagery and 3D Gaussian Splatting scenes. The starting point is a generalization of the Droste effect (both straight and twisted, the latter inspired by Escher's *Print Gallery*) from the Euclidean plane to the sphere, following work by Segerman. We implement these effects in a WebGL pipeline (Three.js + GLSL) and apply them to equirectangular captures from the Marble platform (World Labs).

Rather than exploiting the traditional self-similar recursion of the Droste effect, we repurpose the conformal structure to create *inter-world portals*, regions where one world folds into another. We then propose two extensions: (1) a volumetric formulation where the transformations are applied directly to 3D Gaussian Splat positions in spherical coordinates, and (2) a topological formulation inspired by the cutting-and-gluing of 3-manifolds with varying curvature, drawing on Thurston's geometrization and Perelman's proof of the Poincaré

Conjecture. The project maintains a ludic sensibility, treating the mathematical constructions as both objects of study and instruments of play.

Introduction

1. Motivation

The Droste effect, that uncanny visual recursion in which an image contains a scaled copy of itself, which in turn contains another, ad infinitum, is more than a pictorial curiosity. It is a window into the self-similar structure of conformal dynamics on the plane. When M.C. Escher twisted this recursion in his lithograph *Print Gallery* (1956), he introduced a rotational coupling between scale levels, creating a spiral that has been rigorously decoded by Lenstra and de Smit [1] as the result of a conformal mapping with complex multiplier.

But what happens when we lift these constructions from the flat plane to the sphere? And what happens when, instead of recursing into the same image, we recurse into a *different* one?

These two questions motivate the present work. The first leads us to Möbius transformations on the Riemann sphere and to Henry Segerman's formulation of the spherical Droste effect [2]. The second leads us to the notion of *conformal portals*, regions in visual space where the self-similar tunnel does not fold a world onto itself but instead opens onto another world entirely.

This second question is not merely aesthetic. It touches on the structure of *World Models* [3], internal representations of environments that agents navigate and predict. If a world model is a space, then a portal between worlds is a singularity in that space: a point or region where the geometry fails to be locally Euclidean, or where the atlas of coordinate charts acquires nontrivial transition functions. The mathematical language for such constructions is precisely the language of manifolds, conformal maps, and topological gluing.

2. Context and Scope

This project is part of a study group on World Models at IMPA's Visgraf Lab. One current focus of the group is the Marble platform developed by World Labs (co-founded by Dr. Fei-Fei

Li), which generates navigable 3D worlds from single images and textual prompts. The platform provides a rich source of omnidirectional visual data—both equirectangular images and 3D Gaussian Splatting representations—that serve as the raw material for our experiments.

This report is a *living document*: an initial draft that will be expanded as the project develops. The goals are deliberately open-ended. We aim to:

1. Implement and understand the spherical Droste effect (straight and twisted) via Möbius transformations on the Riemann sphere, applied to 360° imagery from Marble.
2. Reinterpret the Droste recursion not as self-similarity but as an inter-world transition mechanism.
3. Extend the approach from 2D omnidirectional images to 3D Gaussian Splatting representations.
4. Explore the topological and geometric foundations of inter-world connections, including the cutting and gluing of 3-manifolds with different curvatures.
5. Maintain a ludic perspective, treating the mathematical and computational constructions as both research contributions and instruments of play.

3. Report Organization

Section 2 presents the mathematical background: the Droste effect, Escher's twist, Möbius transformations, and the spherical formulation. Section 3 describes our current implementation. Section 4 outlines proposed research directions: conformal portals (4.1), volumetric transformations on Gaussian Splats (4.2), and non-Euclidean topological gluing (4.3).

Mathematical Foundations

1. The Droste Effect and Self-Similarity

The Droste effect is the visual phenomenon where an image contains a (smaller) copy of itself, in a place where such a copy would "naturally" appear (e.g. a picture on a wall, a label on a box, a screen within a screen), and this recursion continues in principle ad infinitum. Mathematically, this is captured by self-similarity under a scaling (plus possible rotation) about a marked point.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain (the "image support"), thought of as the region where the image is defined and $p_0 \in \Omega$ be a marked point (the center of the Droste recursion). An image $I : \Omega \to \mathbb{R}^3$, where $\mathbb{R}^3$ is representing some color space (e.g. RGB values), exhibits the Droste effect when $I(z) = I(\lambda(z - p_0) + p_0)$, for some $\lambda \in \mathbb{C}^*,\ |\lambda| \neq 1$. That is, a complex scaling encoding both dilation and rotation about p_0. The image therefore contains infinitely many scaled copies of itself.

For convenience, we will assume $p_0 = 0$ (by translation). In practice, the image is defined on an annulus $A = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$ with the identification $z \sim \lambda z$, where $|\lambda| = r_2/r_1$. The quotient $A/\sim$ is topologically a torus $T^2 = S^1 \times S^1$, and the visual recursion arises from lifting the image on this torus to its universal cover, which is the plane. More concretely, the Droste image is constructed by tiling the plane with copies of the annular image at scales λ^n for $n \in \mathbb{Z}$.

Applying a logarithm unwraps the Droste annulus to give an infinite vertical strip in $\mathbb{C}$ with width $log(\lambda)$. This vertical strip is periodic with period 2π due to the complete rotation of the annulus. The straight Droste effect is obtained when λ is real, which doesn't introduce a rotation. In this case, all the scaled copies of the annular image become equal strips translated horizontally by $log(\lambda)$. Both periodicities together result in the rectangle of width $log(\lambda)$ and height 2π tiling the entire complex space.

2. Escher's Print Gallery and the Twisted Droste Effect

In *Print Gallery*, Escher introduced a **twist**: each scaled copy is also rotated relative to its parent. To obtain a twisted Droste effect we can start with a straight Droste image and apply a complex scale over its logarithm and then return with the exponential. That is, $I'(z) = I(exp(w\ log(z)))$, $w \in \mathbb{C}$. Notice, however, that $exp(w\ log(z)) = z^w$. Since the original image $I(z)$ exhibits the straight Droste effect, $I'(z) = I(z^w) = I(\lambda\ z^w) = I((\lambda^{1/w}\ z)^w) = I'(\lambda^{1/w}\ z)$.

References

[1] Lenstra, H. W., & de Smit, B. (2003). Artiful Mathematics: The Heritage of M. C. Escher. Nieuw Archief voor Wiskunde.

[2] Schleimer, S. and Segerman, H. (2016). Squares that Look Round: Transforming Spherical Images. arXiv:1605.01396

[3] Ha, D., & Schmidhuber, J. (2018). World Models. arXiv:1803.10122.